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Abstract.   The Gaussian wavefunctional approach is developed in thermofield dynamics. We construct the thermal vacuum wavefunctional, its creation as well as annihilation operators, and accordingly the thermo-particle excited states. For a $(D + 1)$-dimensional scalar field system with an arbitrary potential whose Fourier representation exists in the sense of tempered distributions, we calculate the finite-temperature Gaussian effective potential (FTGEP), one- and two-thermo-particle-state energies. The zero-temperature limit of each of them is just the corresponding result in quantum field theory, and the FTGEP can lead to the same one for some concrete models as calculated by the imaginary-time Green function.

The Gaussian wavefunctional approach (GWFA) [1] has become an important non-perturbative tool of quantum field theory and condensed-matter physics [2, 3] since Stevenson's advocacy of it in 1985 [4]. It is simple, feasible, effective, and can reveal useful non-perturbative information (at least qualitatively). A result from the GWFA can give helpful guidance or be taken as a good starting point for further investigation [5]. Moreover, this approach is often used for some novel investigations or new ideas [6]. So the GWFA was inevitably generalized to finite-temperature field theory, by direct calculation of the partition function, either from first principles or by calculating the real-time or imaginary-time Green functions [7]. However, thermofield dynamics [8], the third formalism of finite-temperature field theory, has its own advantages over the other two formalisms: the closed time path and the imaginary-time formalisms [9]. It can give an answer to some questions which cannot be addressed within the framework of the closed time path formalism, such as the structure of the thermal vacuum, the nature of the Goldstone states, and so on [9]. Also, thermofield dynamics can describe the time development for quantities near equilibrium very naturally [9]. Hence it is worthwhile generalizing the GWFA to thermofield dynamics. In 1986, by considering the $\lambda\phi^4_{(0+1)}$ theory, Roditi gave the quantum-mechanical, finite-temperature Gaussian effective potential (FTGEP) within the framework of thermofield dynamics [10]. Recently, Mishra and Mishra calculated the FTGEP of the $\lambda\phi^4_{(3+1)}$ theory by the combination of the zero-temperature Bogoliubov and the thermal Bogoliubov transformation techniques (TDBTT) [11]. In this paper, we intend to develop Roditi's pioneering work and establish the GWFA of thermofield dynamics (TDGWFA) in $(D + 1)$ dimensions.

We consider a relatively general model with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x - V(\phi_x) \tag{1}$$

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where $\phi_x \equiv \phi(x)$, and the potential $V(\phi_x)$ has a Fourier representation in the sense of tempered distributions [12]. Potentials of many models, such as various polynomial models, sine-Gordon and sinh-Gordon models, have this property. We suppose that the field system (1) is immersed in a thermal reservoir with a fixed temperature $T = 1/k_b\beta$ (k_b represents the Boltzmann constant), i.e. we shall consider a canonical ensemble. (As for the grand canonical ensemble, the generalization is straightforward.) After constructing the thermal vacuum wavefunctional and the thermal particle creation and annihilation operators, we shall calculate the FTGEP, one- and two-thermo-particle energies, and then conclude this paper by a brief discussion.

By the way, thermofield dynamics and the GWFA in quantum field theory will be not introduced in this paper: good expositions of these topics can be found in [8, 9] and [1–3], respectively.

Let us begin with quantum field theory. In the fixed-time functional Schrödinger picture, the canonical conjugate operator to ϕ_x is $\Pi_x \equiv -i\delta/\delta\phi_x$ with the commutator $[\phi_x, \Pi_y] = i\delta(x - y)$. For the convenience of renormalizing the theory of the system (1) later, we shall use the following normal-ordered Hamiltonian density with respect to any normal-ordering mass Q [13]:

$$\mathcal{N}_Q[\mathcal{H}_x] = \tfrac{1}{2}\Pi_x^2 + \tfrac{1}{2}(\nabla\phi_x)^2 - \tfrac{1}{2}I_0[Q^2] + \tfrac{1}{4}Q^2 I_1[Q^2] + \mathcal{N}_Q[V(\phi_x)] \tag{2}$$

where ∇ represents the gradient operator in D-dimensional x-space, $p = |\boldsymbol{p}|$, and

$$I_n[Q^2] = \int \frac{\mathrm{d}^D\boldsymbol{p}}{(2\pi)^D} \frac{\sqrt{p^2 + Q^2}}{(p^2 + Q^2)^n}.$$

As for $\mathcal{N}_Q[V(\phi_x)]$, taking the Fourier transformation (at least in the sense of tempered distributions [12]) and using the Baker–Hausdorff formula, one can have

$$\mathcal{N}_Q[V(\phi_x)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{\sqrt{2\pi}} \tilde{V}(\Omega) \exp(i\Omega\phi_x) \exp\left(\tfrac{1}{4}\Omega^2 I_1[Q^2]\right). \tag{3}$$

As with the GWFA in quantum field theory [1, 3], we take following Gaussian wavefunctional as the quantum vacuum ansatz of the $T = 0$ field system (1):

$$|\varphi\rangle = N_f \exp\left\{-\frac{1}{2}\int_{x,y}(\phi_x - \varphi_x)f_{xy}(\phi_y - \varphi_y)\right\} \tag{4}$$

where φ_x and f_{xy} are variational parameter functions. N_f is some normalization constant, and depends upon f_{xy}. Due to the translational invariance of the vacuum, it is required that $f_{xy} = f(x-y)$. Also, in view of the symmetric form of the vacuum, we assume that $f_{xy} = f_{yx}$. Besides, the inverse f_{xy}^{-1} of f_{xy} must exist. It is easy to show that φ_x is the expectation value of the field operator with respect to the above trial vacuum $|\varphi\rangle$. The creation and the annihilation operators for the vacuum (4) can be constructed [1, 3] as

$$A_f(\boldsymbol{p}) = \left(\frac{1}{2(2\pi)^D f(\boldsymbol{p})}\right)^{1/2} \int_x e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\left[f(\boldsymbol{p})(\phi_x - \varphi_x) + i\Pi_x\right] \tag{5}$$

and

$$A_f^\dagger(\boldsymbol{p}) = \left(\frac{1}{2(2\pi)^D f(\boldsymbol{p})}\right)^{1/2} \int_x e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\left[f(\boldsymbol{p})(\phi_x - \varphi_x) - i\Pi_x\right] \tag{6}$$

respectively, where $f(\boldsymbol{p})$ is the Fourier transform of f_{xy}. Of course, one can check the relations $[A_f(\boldsymbol{p}), A_f^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p}' - \boldsymbol{p})$ and $A_f(\boldsymbol{p})|\varphi\rangle = 0$.

In order to calculate average value of any physical quantity, thermofield dynamics needs the tilde system, the copy of the $T = 0$ field system. Corresponding to the above, the tilde system has the normal-ordered Hamiltonian density

$$\mathcal{N}_Q[\tilde{\mathcal{H}}_x] = \tfrac{1}{2}\tilde{\Pi}_x^2 + \tfrac{1}{2}(\nabla\tilde{\phi}_x)^2 - \tfrac{1}{2}I_0[Q^2] + \tfrac{1}{4}Q^2 I_1[Q^2] + \mathcal{N}_Q[V(\tilde{\phi}_x)] \tag{7}$$

and the quantum vacuum

$$|\tilde{\varphi}\rangle = N_f \exp\left\{-\frac{1}{2}\int_{x,y}(\tilde{\phi}_x - \varphi_x)f_{xy}(\tilde{\phi}_y - \varphi_y)\right\} \tag{8}$$

where $\tilde{\Pi}_x \equiv i\delta/\delta\tilde{\phi}_x$ is conjugate to the tilde field operator $\tilde{\phi}_x$ with the commutator $[\tilde{\phi}_x, \tilde{\Pi}_y] = -i\delta(x-y)$. The appearance of the additional minus in the commutation relation is due to the rule of the tilde operation [8, 9]. Note that it is not necessary to distinguish the ordinary real function and average values of the $T = 0$ system from those of its tilde system. From the rule of the tilde operation, the annihilation and creation operators for the tilde vacuum (8) read

$$\tilde{A}_f(\mathbf{p}) = \left(\frac{1}{2(2\pi)^D f(\mathbf{p})}\right)^{1/2}\int_x e^{i\mathbf{p}\cdot x}\left[f(\mathbf{p})(\tilde{\phi}_x - \varphi_x) - i\tilde{\Pi}_x\right] \tag{9}$$

and

$$\tilde{A}_f^\dagger(\mathbf{p}) = \left(\frac{1}{2(2\pi)^D f(\mathbf{p})}\right)^{1/2}\int_x e^{-i\mathbf{p}\cdot x}\left[f(\mathbf{p})(\tilde{\phi}_x - \varphi_x) + i\tilde{\Pi}_x\right]. \tag{10}$$

Of course, the relations $[\tilde{A}_f(\mathbf{p}), \tilde{A}_f^\dagger(\mathbf{p}')] = \delta(\mathbf{p}' - \mathbf{p})$ and $\tilde{A}_f(\mathbf{p})|\tilde{\varphi}\rangle = 0$ hold. Additionally, each operator of the tilde system commutes with that of the original system.

Based on the $T = 0$ field system and its tilde partner, we now take the following wavefunctional as the thermal vacuum ansatz of the $T \neq 0$ field system:

$$|\varphi, \beta\rangle = \exp\left\{i\int_{x,y}\left[(\phi_x - \varphi_x)g_{xy}(\beta)\tilde{\Pi}_y - \Pi_x g_{xy}(\beta)(\tilde{\phi}_y - \varphi_y)\right]\right\}|\varphi\tilde{\varphi}\rangle \tag{11}$$

where $|\varphi\tilde{\varphi}\rangle$ represents the direct product of $|\varphi\rangle$ and $|\tilde{\varphi}\rangle$, i.e.

$$|\varphi\tilde{\varphi}\rangle = N_f^2 \exp\left\{-\frac{1}{2}\int_{x,y}\left[(\phi_x - \varphi_y)f_{xy}(\phi_y - \varphi_y) + (\tilde{\phi}_x - \varphi_x)f_{xy}(\tilde{\phi}_y - \varphi_y)\right]\right\}. \tag{12}$$

In analogy with f_{xy}, $g_{xy}(\beta) = g_{yx}(\beta) = g(x - y, \beta)$, it also being a temperature-dependent variational parameter function to be determined. When it acts on the thermal vacuum wavefunctional (11), any operator of the original system works only for quantities of the original system itself. So does that of the tilde system. According to [14], one can have (here the Baker–Hausdorff formula is invalid)

$$\exp\left\{i\int_{x,y}\left[(\phi_x - \varphi_x)g_{xy}(\beta)\tilde{\Pi}_y - \Pi_x g_{xy}(\beta)(\tilde{\phi}_y - \varphi_y)\right]\right\}$$

$$= \exp\left\{-\int_{xpy}' \text{th}(g(\mathbf{p}, \beta))\Pi_x(\tilde{\phi}_y - \varphi_y)\right\}$$

$$\times \exp\left\{\int_{xpy}' \ln(\text{ch}(g(\mathbf{p}, \beta)))\left[(\phi_x - \varphi_x)\Pi_y + (\tilde{\phi}_x - \varphi_x)\tilde{\Pi}_y\right]\right\}$$

$$\times \exp\left\{\int_{xpy}' \text{th}(g(\mathbf{p}, \beta))(\phi_x - \varphi_x)\tilde{\Pi}_y\right\}$$

where

$$\int_{xpy}' \equiv i\int_{x,y}\int \frac{d^D p\, e^{i\mathbf{p}\cdot(x-y)}}{(2\pi)^D}$$

and $g(p, \beta)$ is the Fourier transform of the function $g_{xy}(\beta)$. Hence one can write equation (11) as

$$|\varphi, \beta\rangle = N_f^2 \exp\left\{\frac{i}{2} \int'_{xpy} \left[(\phi_x \operatorname{ch}(g(p, \beta)) - \tilde{\phi}_x \operatorname{sh}(g(p, \beta)) - \varphi_x(1 - \operatorname{sh}(g(p, \beta))))\right.\right.$$

$$\times f(p)(\phi_y \operatorname{ch}(g(p, \beta)) - \tilde{\phi}_y \operatorname{sh}(g(p, \beta)) - \varphi_y(1 - \operatorname{sh}(g(p, \beta))))$$

$$+ (\tilde{\phi}_x \operatorname{ch}(g(p, \beta)) - \phi_x \operatorname{sh}(g(p, \beta))$$

$$- \varphi_x(1 - \operatorname{th}(g(p, \beta)) + \operatorname{sh}(g(p, \beta)) \operatorname{th}(g(p, \beta)))) f(p)(\tilde{\phi}_y \operatorname{ch}(g(p, \beta))$$

$$\left.\left. - \phi_y \operatorname{sh}(g(p, \beta)) - \varphi_y(1 - \operatorname{th}(g(p, \beta)) + \operatorname{sh}(g(p, \beta)) \operatorname{th}(g(p, \beta))))\right]\right\}. \quad (13)$$

This wavefunctional is just Gaussian for ϕ or for $\tilde{\phi}$.

For the thermal vacuum (11) or (13), one can construct the following annihilation and creation operators:

$$A_f(p, \beta) = \left(\frac{1}{2(2\pi)^D f(p)}\right)^{1/2} \int_x \left\{f(p)\left[e^{-ip \cdot x} \operatorname{ch}(g(p, \beta))(\phi_x - \varphi_x)\right.\right.$$

$$- e^{ip \cdot x} \operatorname{sh}(g(p, \beta))(\tilde{\phi}_x - \varphi_x)\right] + ie^{-ip \cdot x} \operatorname{ch}(g(p, \beta))\Pi_x$$

$$\left. - ie^{ip \cdot x} \operatorname{sh}(g(p, \beta))\tilde{\Pi}_x\right\} \quad (14)$$

and

$$A_f^\dagger(p, \beta) = \left(\frac{1}{2(2\pi)^D f(p)}\right)^{1/2} \int_x \left\{f(p)\left[e^{ip \cdot x} \operatorname{ch}(g(p, \beta))(\phi_x - \varphi_x)\right.\right.$$

$$- e^{-ip \cdot x} \operatorname{sh}(g(p, \beta))(\tilde{\phi}_x - \varphi_x)\right] - ie^{ip \cdot x} \operatorname{ch}(g(p, \beta))\Pi_x$$

$$\left. + ie^{-ip \cdot x} \operatorname{sh}(g(p, \beta))\tilde{\Pi}_x\right\} \quad (15)$$

with the relations $[A_f(p, \beta), A_f^\dagger(p', \beta)] = \delta(p' - p)$ and $A_f(p, \beta)|\varphi, \beta\rangle = 0$. The thermo-particle creation and annihilation operators of the tilde system can be obtained from (14) and (15) according to the rule of the tilde operation.

Letting the thermo-particle creation operator $A_f^\dagger(p, \beta)$ act on the thermal vacuum (11) or (13), we can construct the one- and two-thermo-particle excited states (in the centre-of-mass frame) as

$$|1\rangle = A_f^\dagger(p, \beta)|\varphi, \beta\rangle \quad (16)$$

and

$$|2\rangle = \int d^D p \, \Sigma(p) A_f^\dagger(p, \beta) A_f^\dagger(-p, \beta)|\varphi, \beta\rangle \quad (17)$$

respectively, where $\Sigma(p)$ is the Fourier transform of the S-wave function of the two thermo-particles with $\Sigma(-p) = \Sigma(p)$. Of course, one can also construct the multi-thermo-particle states and the excited states of the thermal tilde system. We discuss these no further here.

Next we are ready to calculate the FTGEP. From [8], the entropy operator is

$$S = \int dp \left[A_f(p)A_f^\dagger(p) \ln(\operatorname{ch}^2(g(p, \beta))) - A_f^\dagger(p)A_f(p) \ln(\operatorname{sh}^2(g(p, \beta)))\right] \quad (18)$$

and hence, the Helmholtz free-energy operator reads

$$F = \int_x \mathcal{N}_Q[\mathcal{H}_x] - \frac{1}{\beta} S. \quad (19)$$

(Actually $k_B S$ is the entropy). Using functional integration technique and the integral formula $\int_0^\infty (2a/\sqrt{\pi})e^{-a^2 x^2}\, dx = 1$, we obtain the average value

$$\langle \varphi, \beta | \mathcal{N}_Q[V(\phi_x)] | \varphi, \beta \rangle = \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi_x\right) \quad (20)$$

with the notation

$$J_n(g) = \int \frac{dp\, f(p)\, \text{ch}(2g(p, \beta))}{(2\pi)^D (f(p))^{2n}}.$$

Performing some functional integrations leads to the vacuum average value of the free energy:

$$\mathcal{F}[\varphi, g, f; \beta] = \langle \varphi, \beta | F | \varphi, \beta \rangle$$

$$= \frac{1}{4} J_0(g) + \frac{1}{2}(\nabla \varphi_x)^2 + \frac{1}{4} \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{f(p)} \text{ch}(2g(p, \beta)) - \frac{1}{2} I_0(Q^2)$$

$$+ \frac{1}{4} Q^2 I_1(Q^2) + \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi_x\right)$$

$$- \frac{1}{\beta} \int \frac{dp}{(2\pi)^D} \Big[\text{ch}^2(g(p, \beta)) \ln(\text{ch}^2(g(p, \beta)))$$

$$- \text{sh}^2(g(p, \beta)) \ln(\text{sh}^2(g(p, \beta))) \Big] \quad (21)$$

which is a functional of the functions φ_x, g, f. When the field system (1) has a thermal equilibrium with the heat reservoir at $T \neq 0$, its free energy is absolutely minimized. So minimizing the free energy with respect to $f(p)$ (the Fourier transform of the function f_{xy}) and taking $\varphi_x = \text{constant} = \varphi$, we obtain

$$f(p) = f(p, \beta) = \sqrt{p^2 + \mu^2(\varphi, \beta)} \quad (22)$$

where

$$\mu^2(\varphi, \beta) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V^{(2)}\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi\right) \quad (23)$$

with

$$V^{(n)}(z) = \frac{d^n V(z)}{dz^n} = \int_{-\infty}^{\infty} \frac{d\Omega}{\sqrt{2\pi}} (i\Omega)^n \tilde{V}(\Omega) e^{i\Omega z}.$$

Thus, we see that the variational procedure has enforced the function f depend both upon φ and upon β through the function g.

Furthermore, regardless of the relation (22) and independently of the last variational procedure, one can minimize the free energy (21) with respect to $g(p, \beta)$ and obtain

$$g(p, \beta) = g(p, \beta) = \frac{1}{2} \ln\left(\frac{e^{\frac{1}{2}\beta f(p, \beta)} + 1}{e^{\frac{1}{2}\beta f(p, \beta)} - 1}\right). \quad (24)$$

(Here, in the minimized result, we have replaced $f(p)$ by $f(p, \beta)$.) Now the function g is evidently related to β, owing to the variational procedure. This point is similar to [11], but different from [10], where the expression relating θ and β is given artificially (θ there corresponds to g here). Substituting equations (22) and (24) in (21), one has the FTGEP of (1):

$$\mathcal{V}_T(\mu, \beta, \varphi) \equiv \mathcal{F}(\mu, \beta, \varphi) = \mathcal{F}[\varphi = \text{constant}, g_{xy}(\beta), f_{xy}; \beta]\big|_{\mu^2(\varphi, \beta) \to \mu^2}$$

$$= \frac{1}{2} J_0(g) - \frac{\mu^2}{4} J_1(g) - \frac{1}{2} I_0(Q^2) + \frac{1}{4} Q^2 I_1(Q^2)$$

$$+ \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi\right)$$

$$- \frac{1}{\beta} \int \frac{dp}{(2\pi)^D} \Big[\operatorname{ch}^2(g(p, \beta)) \ln(\operatorname{ch}^2(g(p, \beta)))$$

$$- \operatorname{sh}^2(g(p, \beta)) \ln(\operatorname{sh}^2(g(p, \beta))) \Big]. \tag{25}$$

In this equation, '$\to$' means that μ takes the place of $\mu(\varphi, \beta)$. Because of the nature of the minimizing procedure, μ should be chosen from the non-zero root of (23) and two end points of the range $0 < \mu < \infty$ so that $\mathcal{V}_T[\mu, \beta, \varphi]$ is an absolute minimum. Besides, sometimes the non-zero solution of (23) is multi-valued, and so in that case, the suitable root should be decided according to the stability condition

$$\frac{\partial^2 \mathcal{F}[\varphi, g, f; \beta]}{(\partial \mu^2)^2} = \frac{1}{8} J_2(g) \left[1 + \frac{1}{8} J_2(g) \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V^{(4)}\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi\right) \right]$$

$$> 0. \tag{26}$$

For terms comprising $J_n(g)$'s and $I_n(Q^2)$'s in (23), (25) and (26), we can write them as

$$\tfrac{1}{2} J_0(g) - \tfrac{1}{4}\mu^2 J_1(g) - \tfrac{1}{2} I_0(Q^2) + \tfrac{1}{4} Q^2 I_1(Q^2)$$

$$= \tfrac{1}{2}[I_0(\mu^2) - I_0(Q^2)] - \tfrac{1}{4}\mu^2 I_1(\mu^2) + \tfrac{1}{4} Q^2 I_1(Q^2) + C_0 - \tfrac{1}{2}\mu^2 C_2$$

$$J_1(g) - I_1(Q^2) = I_1(\mu^2) - I_1(Q^2) + 2C_1$$

and

$$\frac{1}{\beta} \int \frac{dp}{(2\pi)^D} \Big[\operatorname{ch}^2(g(p, \beta)) \ln(\operatorname{ch}^2(g(p, \beta))) - \operatorname{sh}^2(g(p, \beta)) \ln(\operatorname{sh}^2(g(p, \beta))) \Big]$$

$$= C_0 - \frac{1}{\beta} \int \frac{d^D p}{(2\pi)^D} \ln(1 - e^{-\beta f(p,\beta)})$$

where

$$C_n \equiv \int \frac{d^D p}{(2\pi)^D} \frac{f^{1-n}(p, \beta)}{e^{\beta f(p,\beta)} - 1}.$$

Obviously, it is perhaps hard for C_0, C_1 and C_2 to have simple analytic expressions, but they are all finite. Moreover, for the case of $D < 3$, although the $I_n(\mu^2(\beta, \varphi))$'s and $I_n(Q^2)$'s are divergent, the divergences can all cancelled each other in the above three expressions. So when $D < 3$, equations (23), (25) and (26) do not have any divergence, and accordingly no further renormalization procedures are needed. This indicates that the finite-temperature field theory with (1) is renormalizable for the case of $D < 3$, at least within the framework of the FTGWFA.

Now we are in a position to calculate the one- and two-thermo-particle energies. For thermo-particle excited states, thermofield dynamics uses the following Hamiltonian $\hat{H}$ of the combined system of the $T = 0$ field system and its tilde partner:

$$\mathcal{N}_Q[\hat{H}] \equiv \int_x \{\mathcal{N}_Q[\mathcal{H}_x] - \mathcal{N}_Q[\tilde{\mathcal{H}}_x]\} \tag{27}$$

and each thermo-particle excited state is the eigenstate of $\mathcal{N}_Q[\hat{H}]$. Similar to the calculation for the FTGEP, using a functional integration technique, one can obtain the one-thermo-particle energy

$$m_1(\varphi, \beta) = \frac{\langle 1|\mathcal{N}_Q[\hat{H}]|1\rangle}{\langle 1|1\rangle} = \sqrt{p^2 + \mu^2(\varphi, \beta)} \tag{28}$$

and the two-thermo-particle energy

$$m_2 \equiv \frac{\langle 2|\mathcal{N}_Q[\hat{H}]|2\rangle}{\langle 2|2\rangle}$$

$$= \left(2\int d^D p\, \Sigma^2(p,\beta)f(p,\beta) + \frac{v^{(4)}(\varphi,\beta)}{8(2\pi)^D}\int \frac{d^D p\, \Sigma(p,\beta)}{f(p,\beta)}\int \frac{d^D p\, \Sigma(p,\beta)\,\mathrm{ch}(2g(p,\beta))}{f(p,\beta)}\right)$$

$$\times \left(\int d^D p\,[\Sigma(p,\beta)]^2\right)^{-1} \tag{29}$$

where

$$v^{(n)}(z,\beta) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V^{(n)}\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + z\right). \tag{30}$$

Minimizing m_2 with respect to $\Sigma(p,\beta)$ leads to a second kind of Fredholm integral equation in terms of $\Sigma(p,\beta)$ [15]; solving it, we have

$$\Sigma(p,\beta) = \frac{C}{f(p,\beta)(2f(p,\beta) - m_2)}\left[\mathrm{ch}(2g(p,\beta)) + \int \frac{dp\,\mathrm{ch}^2(2g(p,\beta))}{f^2(p,\beta)(2f(p,\beta) - m_2)}\right.$$

$$\left.\times \left(1 + \frac{v^4(\varphi,\beta)}{16(2\pi)^D}\int \frac{dp\,\mathrm{ch}(2g(p,\beta))}{f^2(p,\beta)(2f(p,\beta) - m_2)}\right)^{-1}\right] \tag{31}$$

where C is some normalization constant. Substituting the last expression in (29) and noting the normalization of C, one can compute the energy m_2. From equation (28), one can see that $\mu(\varphi,\beta)$ is just the mass of one thermal particle. Thus, equation (23) can be used to calculate the mass of a single thermo-particle. Moreover, the two terms in (29) can be interpreted as the kinetic energy of the two constituent thermal particles and their interacting energy, respectively. One can perhaps dicuss thermo-particle bound states and scattering states with the help of (29) [3, Lu *et al* 1993].

In this paper, we have developed the GWFA of thermofield dynamics in $(D+1)$ dimensions, and calculated the FTGEP, one- and two-thermo-particle-state energies of the system (1). Equations (25), (28) and (29) with some relevant relations are convenient for an investigation of any concrete model involved in the class of (1) (including some bosonized models in condensed-matter physics [16]). When equation (25) is applied to $\lambda\phi^4$ and ϕ^6 models, one can obtain the same, corresponding, results as in [7, Roditi, Okopinska, Hajj and Stevenson, and Cea]. Again, contrasting the results in [17] with those here, and noting the above-mentioned analysis about the $J_n(g)$'s and $I_n(Q^2)$'s, one can have the replacement rule between the FTGEP and GEP indicated in [7] ($J_n(g) \to I_n(\mu^2(\varphi))$). Still, the $T = 0$ limit of each of (25), (28) and (29) is identical with that in [17]. Moreover, employing equations (5), (6), (9) and (10), one can find that the exponential factor in (11) is just the field-operator form of a thermal Bogoliubov transformation, and one can therefore conclude that the TDGWFA is equivalent to the TDBTT. In fact, when $I_n(Q^2)$'s are deleted, equation (25) can give the non-renormalized equation [11, equation (33)]. Besides, if a non-uniform background field is considered, the term $\frac{1}{2}(\nabla\varphi_x)^2$ will be added to (25) with φ_x replacing φ, and then by

$$\frac{\partial(\mathcal{V}_T[\mu,\beta,\varphi_x])}{\partial\varphi_x} = 0$$

one can have the following equation:

$$\nabla^2\varphi_x - \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\frac{1}{4}\alpha^2} V^{(1)}\left(\tfrac{1}{2}\alpha\sqrt{J_1(g) - I_1(Q^2)} + \varphi_x\right) = 0 \tag{32}$$

the $T = 0$ limit of which is identical with that of [17]. This equation can be used for considering quantum thermal solitons. Based on the thermal soliton solution, equations (28) and (29) can perhaps give the excited-state results for thermo-solitons. Of course, when the non-uniform background field is considered, the discussion about the difference between the BTT and GWFA in [17] is valid for the finite-temperature case. Finally, the GWFA of thermofield dynamics can be generalized to the time-dependent case to investigate the dynamics of statistical mechanical systems.

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